A Geometric Approach to Inelastic Collapse*

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Abstract

We show in this note how to interpret logarithmic spiral tilings as one-dimensional particle systems undergoing inelastic collapse. By deforming the spirals appropriately, we can simulate collisions among particles with distinct or varying coefficients of restitution. Our geometric constructions provide a strikingly simple illustration of a widely studied phenomenon in the physics of dissipative gases: the collapse of inelastic particles.

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1 Introduction

Collisions in a granular gas preserve momentum but not kinetic energy. Interactions are dissipative, with the velocities of two colliding particles governed by a stochastic matrix $\left(\begin{smallmatrix} p & q \\ q & p \end{smallmatrix}\right)$, for $p \leq 1/2$. When the coefficient of restitution, defined as $r = 1 - 2p$, is less than 1, the collisions are inelastic and the particles may collapse to a single point in a finite amount of time: this intriguing phenomenon of *inelastic collapse* was first investigated in one dimension by Bernu & Mazighi [2] and McNamara & Young [6]. Further studies and extensions to a larger number n of particles were given in [1, 2, 3, 4, 5, 6, 7, 8]. In the case $n = 3$, inelastic collapse requires $r < 7 - 4\sqrt{3}$ [4, 6, 7], while in general the requirement is that $n \gtrsim 2(\ln 2)/(1 - r)$. Matching constructions for large n exist but entail intricate eigenvalue estimates [1, 2]. We rederive these bounds by simple geometric means, and we also extend them to other types of collisions. Our particle systems are derived from one-dimensional projections of spiral tilings of a disk (see §2). Using different spirals allows the presence of particles with different coefficients of restitution (see §3). The notable feature of our arguments is to be entirely geometric.

2 The Inelastic Collapse of Identical Particles

We describe the dynamics of n identical particles moving towards the center of a disk and colliding along the way. The one-dimensional system is derived by projection to a line. We begin with the geometry of the system, which is a quadrilaterial tiling of the complex unit disk by logarithmic spirals.

2.1 Spiral tilings

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2.2 Spiral tilings

Fix $0 < \lambda_o < 1$ and let $\mathcal{C}_\alpha = \left\{ \lambda_o^{|\varphi-\alpha|} e^{i\varphi} \,|\, \varphi \in \mathbb{R} \right\}$. The curve $\mathcal{C}_\alpha$ consists of two logarithmic spirals running clockwise and counterclockwise from the point $e^{i\alpha}$. The family $\{\mathcal{C}_\alpha\}_{0 \le \alpha < 2\pi}$ forms two foliations of the unit complex disk $\mathcal{D}$ (minus the origin). Whereas no pair of spirals going in the same direction meet, the other pairs intersect infinitely often along the diameter bisecting their starting points. Fix an integer $n > 2$ and write $\theta = \pi/n$. We rectify the spiral $\mathcal{C}_\alpha$ by creating the vertices $\lambda_o^{|k\theta-\alpha|} e^{ik\theta}$ for all $k \in \mathbb{Z}$; then we join consecutive pairs by straightline segments, which produces the polygonal spiral $\mathcal{C}_\alpha^R$ in Figure 1(i).

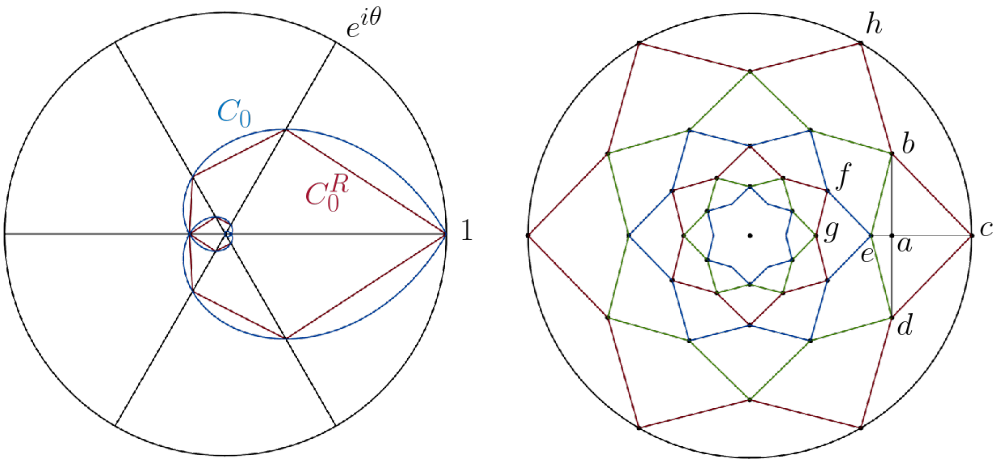

Figure 1 (i) The spirals $\mathcal{C}_\alpha$ and $\mathcal{C}_\alpha^R$, for $\alpha = 0$ and $\theta = \pi/3$; (ii) an (n, λ)-tiling for a system of $2n = 12$ colliding particles.

The collection of polygonal curves $\left\{ \mathcal{C}_{2j\theta}^R \,|\, 0 \le j < n \right\}$ forms an infinite sequence of nested concentric similar $2n$-gons $P_k := \lambda e^{i\theta} P_{k-1}$, where $\lambda = \lambda_o^\theta$ and P_0 is the outer "star" shown in Figure 1(ii): its vertices $e^{il\theta}\lambda^{(1-(-1)^l)/2}$ run in counterclockwise order ($0 \le l < 2n$). To ensure that the shape is indeed a star, every other vertex of P_0 needs to be reflex, which requires that $\lambda < \cos\theta$. This partitions the polygon P_0 into an infinite collection of similar convex quadrilaterals, which forms an (n, λ)-*tiling*. We define the *fundamental ratio* $\rho := ae/ac$ of the (n, λ)-tiling and justify its name by noting that it is independent of the polygon P_k used to define it. Referring to Figure 1(ii), we observe that $ac = 1 - \lambda\cos\theta$ and $ae = \lambda\cos\theta - \lambda^2$ and that, for any $0 < \lambda < \cos\theta$,

$$\rho = \frac{\lambda(\cos\theta - \lambda)}{1 - \lambda\cos\theta} \qquad \text{and} \qquad 0 < \rho < 1\,. \tag{1}$$

2.3 Particles traveling in a disk

Place two particles at each one of the n outer vertices of P_0 and set them in motion along the two incident edges with a speed equal to bc. We show below that the particles will zigzag toward the center (as in the trajectory $c, b, e, f, g, \ldots$) provided that the coefficient of restitution r is equal to $\rho < 1$, where $r = 1 - 2p$; recall that, whenever two particles with velocities $u, v \in \mathbb{C}$ collide, they bounce away from each other and update their velocities as follows:

$$\begin{pmatrix} u \\ v \end{pmatrix} \leftarrow \begin{pmatrix} p & q \\ q & p \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix};$$

where $0 < p < q < 1$ and $p + q = 1$.

▶ **Lemma 2.1.** *The $2n$ particles travel along the edges of the tiling through pairwise collisions if and only if the fundamental ratio ρ is equal to the coefficient of restitution r. If each particle spends one unit of time on the boundary ∂P_0, then it travels on ∂P_k for a duration of δ^k, where $\delta = \lambda^2/\rho$. The total travel time is bounded if and only if $\lambda < \frac{1}{\cos\theta} - \tan\theta$, in which case it is equal to $1/(1-\delta)$.*

Proof. For convenience, we tilt the tiling by θ to put b and f on the X-axis (Figure 2). Two particles travel from c and h to b with velocity u and v respectively. The first one bounces at b and proceeds with velocity $u' = pu + qv$. Since $u_x = v_x$ and $u_y = -v_y$, we have $u'_x = u_x$ and $u'_y = -ru_y$; therefore $|slope(u')| = r|slope(u)|$. By similarity, bc and ef are parallel; hence $|slope(u')| = r|slope(ef)|$. The consistency of the particle collision with the tiling means that u' should be parallel to the segment be. The condition thus becomes $|slope(be)| = r|slope(ef)|$; hence $r = mf/mb = \rho$.

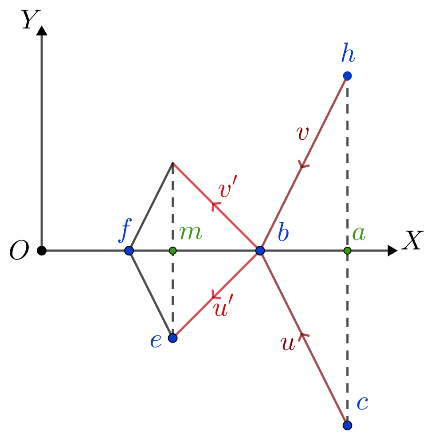

Figure 2 How colliding particles follow the edges of the (n, λ)-tiling. The coefficient of restitution must be equal to the ratio $\rho = mf/mb$.

If the particle travels from c to b in one unit of time, then $u_y = ac$ and $u'_y = -ru_y = -rac$. It follows that the time δ for the particle to bounce from b to e is equal to $me/|u'_y| = \frac{1}{r}me/ac = \lambda^2/r$. More generally, δ is the ratio between the time spent on be and that spent on cb. By symmetry, the same ratio δ holds between the travel times along any two consecutive edges on the trajectory. This follows from the fact that the travel time along an edge is itself a ratio length/speed and that, from one boundary ∂P_k to the next, ∂P_{k+1}, the ratio between consecutive lengths is independent of k and the same is true of consecutive speeds. This implies a travel time of δ^k on ∂P_k. Convergence implies that $\delta < 1$, which, by (1), means that λ must be less than the smaller root of $\lambda^2 \cos\theta - 2\lambda + \cos\theta$ (since the larger one exceeds 1). This gives us the inequality $\lambda < (1 - \sin\theta)/\cos\theta$. Note that this condition is not implied by the previous requirement that $0 < \lambda < \cos\theta$. ◀

By (1), setting $r = \rho$ for any $\lambda < \cos\theta$ produces a valid particle system traveling inward through the (n, λ)-tiling. Of course, the interesting question is whether this holds for *any* value of the coefficient of restitution. We address this issue below in the context of one-dimensional systems.

2.4 One-dimensional collapse

The real parts of the $2n$ particles' positions in the unit disk $\mathcal{D}$ describe a one-dimensional particle system. To see why, it is useful to distinguish between the *positive* particles, those numbered $1, \ldots, n$ counterclockwise around $\mathcal{D}$, from the others, the *negative* particles. The name comes from the fact that the positive (resp. negative) particles always remain in the upper (resp. lower) complex halfplane. Each positive particle j is naturally paired with the negative particle $2n + 1 - j$, since their trajectories are conjugate. Particles can only collide with other particles of the same sign or with their conjugates; in the latter case, the collision does not alter the motion along the real axis. All the other collisions occur in conjugate pairs. This shows that the real-axis motion of the positive particles alone constitutes a bona fide collision system over n particles with the same coefficient of restitution.

▶ **Theorem 2.2.** *Fix any integer $n > 2$, and write $\theta = \pi/n$ and $r_0 = (1 - \sin\theta)/(1 + \sin\theta)$. Given any positive coefficient of restitution $r \leq r_0$, there is a scaling factor λ such that the line projection of the (n, λ)-tiling forms the trajectory of a one-dimensional n-particle system exhibiting inelastic collapse. The collapse time is $r/\big(r - \lambda^2\big)$ for any $r < r_0$ and $\lambda = q\cos\theta - (q^2\cos^2\theta - r)^{1/2}$, where $q = (1 + r)/2$.*

Proof. Setting $r = \rho$ in (1) yields the quadratic equation

$$\lambda^2 - 2q(\cos\theta)\lambda + r = 0; \tag{2}$$

hence $\lambda = q\cos\theta \pm \sqrt{q^2\cos^2\theta - r}$. The roots need to be real; hence $\sin\theta \leq p/q$ or, equivalently, $r \leq r_0$. We verify that $0 < \lambda < \cos\theta$, as required of a valid (n, λ)-tiling, which is a consequence of $\sqrt{q^2\cos^2\theta - r} < p\cos\theta$. By Lemma 2.1, the collapse time is infinite if $\delta = \lambda^2/r \geq 1$ and equal to $\sum_{k\geq 0}\delta^k = 1/(1 - \delta) = r/(r - \lambda^2)$ if $\delta < 1$. The smaller root of (2), if strictly smaller, always satisfies the latter condition while the larger one never does. This follows from the fact that $\lambda_-\lambda_+ = r$, $q\cos\theta \geq \sqrt{r}$, and $\lambda_+ \geq q\cos\theta$; hence $\lambda_+^2 \geq r$. ◀

In our construction, the upper bound on the coefficient of restitution is $(1-\sin\theta)/(1+\sin\theta)$. As n goes to infinity, this gives us $n \gtrsim 2\pi/(1 - r)$, which matches the bounds from [1, 2]. For $n = 3$, our construction rediscovers the classic bound of $7 - 4\sqrt{3}$ [4, 6, 7].

3 Distinct Coefficients of Restitution

Our construction does not require a fixed scaling λ. Instead of placing the vertices on circles of radius λ^k for $k \geq 0$, we can use an arbitrary decreasing radius sequence $(\lambda_k)_{k\geq 0}$, with $\lambda_0 = 1$. We assign a coefficient restitution r_k for the collisions at radius λ_k; the dependency on k might reflect a gain or loss of elasticity after repeated collisions. For notational convenience, let $p = (1 - r_1)/2$, $\lambda = \lambda_1$, and $\mu = \lambda_2$. By reference to Figure 3, we now kick a particle from a to b with velocity $u = b - a$ (using complex numbers), and one from c to b with velocity $v = b - c$. Post-collision, the first particle travels from b to d with velocity $u' = pu + (1 - p)v = \sigma_1(d - b)$, for some $\sigma_1 > 0$; hence $b - c + p(c - a) = \sigma_1(d - b)$. Since $a = 1$, $b = \lambda e^{i\theta}$, $c = e^{2i\theta}$, and $d = \mu$, we divide the equation by $e^{i\theta}$ and find that

$$\lambda - e^{i\theta} + 2pi\sin\theta = \sigma_1\big(\mu e^{-i\theta} - \lambda\big);$$

therefore, $\lambda - \cos\theta = \sigma_1(\mu\cos\theta - \lambda)$ and $r_1 = \sigma_1\mu$. More generally, for $k > 0$, we replace λ and μ by λ_k and λ_{k+1}, respectively, and we scale the relations by λ_{k-1}:

$$\sigma_k = \frac{\lambda_{k-1}\cos\theta - \lambda_k}{\lambda_k - \lambda_{k+1}\cos\theta} \qquad \text{and} \qquad r_k = \frac{\cos\theta - \lambda_k/\lambda_{k-1}}{\lambda_k/\lambda_{k+1} - \cos\theta}\,. \tag{3}$$

Of course, we retrieve the relation $r = \rho$ in (1) in the case $\lambda_k = \lambda^k$ corresponding to having fixed coefficients of restitution.

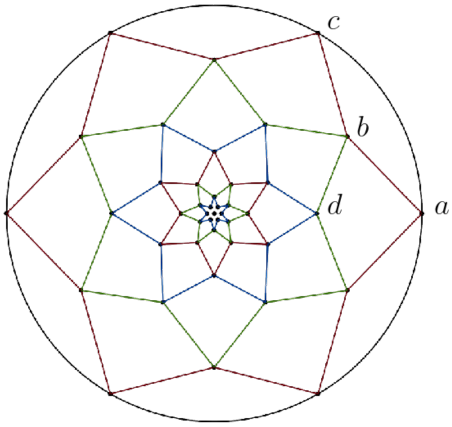

Figure 3 An irregular tiling.

3.1 Finite-time inelastic collapse

From the relation $u' = \sigma_1(d - b)$, we see that the time spent crossing bd is precisely $1/\sigma_1$. More generally, $1/\sigma_k$ is the time spent on the $(k+1)$-st star polygon, given a unit travel time on the previous polygon. It follows that the total travel duration is the sum of all the products of the form $1/\sigma_1 \cdots \sigma_k$, which is

$$1 + \sum_{k=1}^{\infty} \prod_{j=1}^{k} \frac{\lambda_j - \lambda_{j+1}\cos\theta}{\lambda_{j-1}\cos\theta - \lambda_j}\,. \tag{4}$$

By projection onto the real line, finite-time inelastic collapse is guaranteed if

$$\lambda_{k+1} \geq \frac{1+c}{\cos\theta}\lambda_k - c\lambda_{k-1},$$

for some fixed $c < 1$. Again, we can check that, if $\lambda_k = \lambda^k$, then bounded travel time means that $\lambda < \frac{1}{\cos\theta} - \tan\theta$, as claimed in Lemma 2.1.

3.2 Red-blue particles

Consider two species of particles, blue and red. The blue particles collide together with the coefficient of restitution r_1 and the same is true of the red ones. Particles of different colors, however, collide with the coefficient r_2. Arrange the particles as usual, with the sequence blue, blue, red, red, blue, blue, red, red, etc. Set the scaling factor $\lambda_k = \mu^j$ if $k = 2j$, and $\lambda_k = \lambda\mu^j$ if $k = 2j+1$. By (3), we choose

$$r_1 = \frac{\mu(\cos\theta - \lambda)}{\lambda - \mu\cos\theta} \qquad \text{and} \qquad r_2 = \frac{\lambda\cos\theta - \mu}{1 - \lambda\cos\theta}\,.$$

Each factor in (4) is of the form

$$\frac{\lambda_j - \lambda_{j+1}\cos\theta}{\lambda_{j-1}\cos\theta - \lambda_j} = \begin{cases} \mu(1-\lambda\cos\theta)/(\lambda\cos\theta-\mu) = \mu/r_2 & \text{if } j \text{ is even} \\ (\lambda - \mu\cos\theta)/(\cos\theta - \lambda) = \mu/r_1 & \text{else}. \end{cases}$$

The travel time is finite if $\mu^2 < r_1 r_2$, which is

$$\mu(\lambda - \mu\cos\theta)(1-\lambda\cos\theta) < (\cos\theta - \lambda)(\lambda\cos\theta - \mu).$$

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