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A DECISION PROCEDURE FOR OPTIMAL POLYHEDRON PARTITIONING *

Bernard CHAZELLE

Department of Computer Science, Carnegie-Mellon University, Pittsburgh, PA 15213, U.S.A.

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1. Introduction

The problem of decomposing a geometric structure into simpler components has been extensively studied [1,2,4,6,8] for a number of reasons which, besides theoretical interest, include many practical applications in graphics, pattern recognition, tool design, etc. [1,9]. The problem, as most commonly encountered, can be stated as follows:

Given a polygon (resp. polyhedron) P, *what is the smallest set of non-overlapping convex polygons (resp. polyhedra), whose union is exactly* P?

In two dimensions, the problem has the intriguing property of being solvable in polynomial time, yet of being NP-hard, if *holes* are allowed into the polygons [6]. In three dimensions, because of the possibly large size of the output, the problem has a quadratic lower bound on its complexity [2]; furthermore it has been shown to be NP-hard [6]. It is unknown, however, whether the problem is in NP, and actually the absence of trivial enumerative procedures raises the issue of decidability altogether. We prove in this paper that the problem is indeed decidable (as intuition reasonably suggests), by using a combinatorial argument based on Tarski's fundamental result on the decidability of algebra [11].

Let P be a three-dimensional polyhedron with N vertices, and let S denote a partition of P into a minimum number of convex parts. S is called a *minimal convex partition* of P, or MCP as a shorthand. Informally, S is a collection of non-overlapping convex polyhedra (called *parts*), whose union is P. More precisely, we regard S as a *complex*, i.e., as a set of pairwise disjoint polyhedra sharing common faces, edges and vertices [5]. We will assume all complexes considered here to have a *normal* form. By this, we mean that each vertex should be of degree at least 3, and if two faces F and G of two distinct convex parts intersect in a polygon H, the 3 polygons F, G and H are identical. Furthermore, we require that any pair of parts should have at most one common face. These assumptions allow us to define the following sets without ambiguity: V, E, F are respectively the set of all vertices, edges and faces involved in S (each of them occurring exactly once in its respective set). A vertex, an edge or a face is said to be *external* if it lies entirely on the boundary of P (note that it does not have to be a vertex, edge or face of P). We designate the sets of external vertices, edges or faces by V_e, E_e, F_e, respectively. Similarly, all the other vertices, edges and faces involved in S are called *internal*, and form the sets V_i, E_i, F_i. Finally, we define V_0, E_0, F_0 respectively as the sets of vertices, edges and faces of P. Note

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that E_0 (resp. F_0) may not be a subset of E_e (resp. F_e). For simplicity, we will use the same notation to designate a set and its cardinality, as long as there is no ambiguity. We may then write

$$V_0 = N, \qquad V = V_e + V_i, \qquad E = E_e + E_i,$$
$$F = F_e + F_i.$$

2. The decidability argument

To prove that an MCP of P is effectively computable, we first establish an upper bound on the maximum number of vertices present in an MCP. Then we define the *scheme* of a partition as a topological description of the corresponding complex, and we proceed with a combinatorial enumeration of all possible schemes. Each scheme must then be tested for *realizability*, that is, it must be determined to correspond or not to an actual convex partition. A system of equations involving the coordinates of the vertices can be set up to express realizability, from which the problem becomes purely algebraic. Finally, we obtain an MCP as any realizable scheme of minimum cardinality. Before proceeding with a description of the decision procedure, we need some preliminary results. Our first task will be to show that the number of internal faces is polynomial in the number of convex parts. Note that this is trivially false if the convexity requirement is relaxed.

Lemma 2.1. $F_i \leqslant \frac{1}{2}S(S-1)$.

Proof. We identify the polyhedra and the internal faces in S with respectively the nodes and the edges of a graph. An edge will connect two nodes if and only if the corresponding polyhedra share a common face. Since S is given in a normal form, each internal face corresponds to a distinct edge in the graph and there are no multiple edges, which establishes the result. □

We next exhibit an upper bound on the total number of vertices in S.

Lemma 2.2. *If* S *is an MCP of* P, *then* $V = O(V_0^5)$.

Proof. We first ensure that the boundary of P has been triangulated, i.e., each face is a triangle. This can be done without adding new vertices to V_0. Since the boundary of P has the structure of a connected planar graph, we have the well-known relations

$$E_0, F_0 = O(V_0). \tag{1}$$

Considering the complex S, we observe that each edge has two endpoints and each endpoint is adjacent to at least 3 edges; therefore $V \leqslant \frac{2}{3}E$. Euler's formula, generalized to complexes [5], yields $V = S + 1 + E - F$, which gives

$$V \leqslant 2(F - S - 1). \tag{2}$$

Like the faces of P, F_e forms a planar graph embedded on the boundary of P. Moreover, the faces of F_e are subfaces of F_0, i.e., each face of F_e lies entirely within some face of F_0 (recall that the faces of F_0 have been made into triangles). Let h be the maximum number of faces of F_e lying inside a single face of F_0. Inside a triangle of F_0 lies either no edge at all or a planar graph whose edges lie on the boundary of some internal faces. Since internal faces are convex, each contributes at most one edge to this graph, and relation (1), applied to this planar graph, shows that $h = O(F_i)$, therefore $F_e = O(F_0 F_i)$. Since $F = F_e + F_i$, we derive from (2) that $V = O(F_0 F_i)$, and from Lemma 1 that $V = O(F_0 S^2)$. It has been shown in [2] that it is always possible to partition P into $O(E_0^2)$ convex pieces; therefore, since S is an optimal partition, its cardinality satisfies $S = O(E_0^2)$. This shows that $V = O(F_0 E_0^4)$, which combined with relation (1) establishes the lemma. □

This result enables us to try out all possible complexes involving $O(V_0^5)$ vertices, and keep only the MCP's. The method is, in a sense, a generalization of the procedure given in [5] for enumerating combinatorial types of d-polytopes. We define the *scheme* of S, or of any complex in general, as an enumeration, for each polyhedron in the complex, of all its vertices, edges and faces (edges and faces are given by the subsets of vertices which they involve). Giving the location of the vertices of P along with the scheme of S specifies the partition of P exactly. Let W be a subset of V, or a set of subsets of V, or a set of sets of subsets of V (and so forth); we define L(W) to be the subset of V which contains exactly the vertices involved in W. Throughout, sets are to be taken in the usual sense, i.e., not as lists or multisets, thus without notions of order or repetition.

We next introduce the notion of abstract scheme. An *abstract scheme* for P is a family A of nonempty sets $A_1, \ldots, A_p$. Each A_i is a set of subsets of a set $V = \{v_1, \ldots, v_k\}$ with $N = V_0 \leqslant k$. The first N elements $v_1, \ldots, v_N$ correspond to the N vertices of P and have pre-specified coordinates, whereas each v_i, for $i > N$, represents an added (*Steiner*) point of the partition, and is assigned three variables x_i, y_i, z_i, corresponding to its coordinates. Moreover, we require that $L(A) = V$ and that Condition (ST) be satisfied for all W in $\{A_1, \ldots, A_p\}$.

Condition (ST).
(1) L(W) *does not belong to* W.
(2) *If* x *belongs to* L(W), *then the singleton* {x} *belongs to* W.

An abstract scheme of P describes a tentative partition of P. Each A_j corresponds to a polyhedron of the partition, and the requirements can be interpreted as follows: (1) V indeed represents the set of all vertices, (2) all the vertices of A_j cannot lie on the same face (clause (1) of Condition (ST)), and (3) all the vertices of A_j involved in edges or faces are explicitly listed as vertices (clause (2) of Condition (ST)). A scheme of P is an abstract scheme for P. Conversely we want to investigate the conditions for an abstract scheme to be *realizable*, that is, to be isomorphic to the scheme of a convex partition of P. The idea is to list out all possible abstract schemes which can lead to an MCP, and keep the optimal ones among the subset of those which are realizable. We have the following characterization of realizability.

Lemma 2.3. *The abstract scheme* A *for a nonconvex polyhedron* P *is realizable if and only if it is possible to find reals* (x_i, y_i, z_i) *for* $i = N+1, \ldots, k$, *such that the following propositions are true*:
(A) *For every* W *in* $\{A_1, \ldots, A_p\}$,
(1) *For every nonempty subset* Z *of* L(W), *propositions* (a) *and* (b) *are equivalent*:
(a) Z *belongs to* W.
(b) *There exist reals* a, b, c *such that*

$$ax_i + by_i + cz_i \begin{cases} = 1 & \textit{if } v_i \textit{ belongs to } Z, \\ > 1 & \textit{if } v_i \textit{ belongs to } L(W)\setminus Z. \end{cases}$$

(2) *Any point in the convex hull of* L(W) *lies in* P.
(3) *For every* $A_i \neq W$, *no point in the convex hull of* $L(A_i)$ *lies strictly in the convex hull of* L(W).
(B) *For every point* M *in* P, *there exists an index* i, $1 \leqslant i \leqslant p$, *such that* M *lies inside the convex hull of* $L(A_i)$.

Proof. Proposition (A) ensures that (1) each A_i is the scheme of a convex polyhedron, (2) this polyhedron lies in P, and (3) the polyhedra are pairwise disjoint. Finally, proposition (B) guarantees that the set of polyhedra covers P entirely. Thus it suffices to show that (1) holds if and only if W is the scheme of a convex polyhedron with vertices L(W). (b) expresses the fact that all the vertices of Z lie on a plane while the others do not, yet lie on the same side. (a) ⇒ (b) means that all the vertices, edges and faces expressed in W lie on the convex hull of L(W). Also, the requirement

$$[\text{v belongs to } L(W) \;\Rightarrow\; \{v\} \text{ belongs to } W]$$

guarantees that all the vertices of L(W) are vertices of the convex hull of L(W). Conversely, (b) ⇒ (a) shows that W lists exactly all vertices, edges and faces of the convex hull of L(W). □

We can now prove our main claim.

Theorem 2.4. *The problem of partitioning any polyhedron into a minimum number of convex parts is decidable.*

Proof. The crux of the argument is a fundamental result by Tarski on the decidability of first-order sentences in the field of real numbers (see [11], but also [3,7,10]). This result asserts that every statement in elementary algebra (theory of real numbers) containing no free variables is effectively decidable. It is easy to see that the characterization of Lemma 2.3 can be expressed as such a logical statement. Any point inside the convex hull of a set of vertices can be expressed as a linear combination of the vertices with positive coefficients summing up to at most one. In order to express the inclusion of a point M in P, we might first partition P into a (near-optimal) number of convex parts (using the heuristic of [2], for example), then OR the statements relative to the inclusion of M in each part. Having proved that any abstract scheme for P can be determined to be realizable or not, we may simply list all possible abstract schemes involving $O(V_0^5)$ vertices. Lemma 2.2 shows that if we list them by increasing number of families A_i, the first scheme found realizable will provide an MCP of P. □

The decision procedure described above does not provide any practical means for computing MCP's. We have made no attempt to analyze its performance, yet from [7] we are led to believe that $2^{2^{O(N)}}$ might be an upper bound on its execution time.

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